Avoided crossings: Curvature distribution and behavior of eigenfunctions of pseudointegrable and chaotic billiards

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We perform numerical studies on the curvature distribution of classically pseudointegrable and chaotic quantum billiards. This curvature distribution is Gaussian orthogonal ensemble–like for both systems and corresponds to linear level repulsion. The billiards show nonuniversal behavior at low curvatures. As an aside, avoided crossings of three levels are investigated for both systems. Strong interaction between the levels E_i and E_{i+2} is shown. Eigenvalues and eigenfunctions are calculated very precisely by a grid method with an asymmetric sparse matrix.

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I. INTRODUCTION

Up to now investigations of pseudointegrable billiard systems [1,2] and their intermediate position between integrable and chaotic systems were mainly concerned with the statistical properties of their energy spectra [3–8] or with periodic orbit theory [9–11]. But there is another approach to quantum chaotic systems in terms of level dynamics [12–16] in which the behavior of the energy spectrum of a parameter dependent Hamiltonian is analyzed (the "eigenvalue motion"). Keeping in mind the phenomenon of level repulsion and avoided level crossings (ACs), Gaspard *et al.* introduced the *curvature distribution* as a possibility to characterize quantum chaotic systems [14]. As level repulsion is observed in pseudointegrable systems [1], too, in this paper numerical studies on the curvature distribution of pseudointegrable and chaotic billiards are performed. Additionally the behavior of eigenfunctions in the vicinity of avoided crossings is investigated [23].

II. THEORETICAL BACKGROUND

In contrast to integrable Hamiltonian systems, for which the classical motion in phase space takes place on a torus (genus $g = 1$), the phase space trajectories of pseudointegrable systems are confined to an invariant manifold of more complicated topological features ($g > 1$) [1,2]. Typical pseudointegrable billiards are the polygonal billiards. For a billiard system of the shape of n rectangular steps the invariant manifold has genus $g = n$. [1,2,8] [see small geometries in Figs. 1(a)–1(c)].

To study the parametric properties of the spectra of billiard systems, one can change the geometry, e.g., the radius of the circular obstacle in the Sinai billiard. Although the original studies about level dynamics apply for Hamiltonian systems of the form $H = H_0 + \tau V$, quantum billiards, which cannot be represented in this form, can be treated in the same way [17,18]. The second derivative of the parameter dependent eigenvalues of H with respect to the parameter τ yields the level curvature

$$K_i(\tau) := \ddot{E}_i(\tau), \tag{1}$$

where E_i belongs to a uniform energy spectrum $\{E_i\}$. For systems with Hamiltonian $H = H_0 + \tau V$, the *curvature distribution density* $P(K)$ can be introduced [14] as

$$P(K) := \frac{d(\lim_{\mathcal{N}\{E_i\}\to\infty} \mathcal{N}\{E_i(\tau) : K_i(\tau) < K\})}{dK}. \tag{2}$$

Here $\mathcal{N}$ denotes the number of elements in a set. $P(K)dK$ is the probability of finding $K_i(\tau)$ within the interval $[K, K + dK]$. Level dynamics can be translated into a classically integrable system with infinitely many degrees of freedom. Applying methods of statistical mechanics to the resulting "eigenvalue gas" for high values of $|K|$ it can be derived that for Hamiltonians taken from the Gaussian orthogonal, unitary, or symplectic ensemble of random matrices (GOE,GUE,GSE) [19],

$$P(K) \sim |K|^{-\nu-2}, \tag{3}$$

where $\nu = 1, 2, 4$ for the GOE, GUE, and GSE [14,16].

The exponent ν is related to the behavior of the spacing distribution $p(S)$ of the GOE, GUE, and GSE spectra for $S \to 0$, which is [20,21]

$$p(S) \approx S^{\nu}. \tag{4}$$

For the Bunimovich stadium numerical studies have shown close agreement with the prediction for the GOE Hamiltonian $P(|K|) \sim |K|^{-3}$ [22], which corresponds to the observed universality of GOE fluctuations in the energy spectra of chaotic systems. Other quantum chaotic systems have been considered in [15] and have proven to show the same universal behavior at high curvatures.

As it is known that for pseudointegrable billiards the degree of level repulsion is $\nu = 1$, too, the same universal behavior for the tail of the curvature distribution can be expected. However, solitonlike structures in the parametric motion of the eigenlevels [see straight lines in Fig. 2], which can be associated with scarred eigenfunctions, cause nonuniversal behavior of the curvature distribution at low curvatures [15,22]. For this reason, it is particularly interesting to see whether significant differences between the pseudointegrable systems and the chaotic ones show up in the region of low curvature.

III. NUMERICAL RESULTS

We make use of a computational method for solving partial differential equations, especially eigenvalue problems [23]. This method allows high-precision calculations

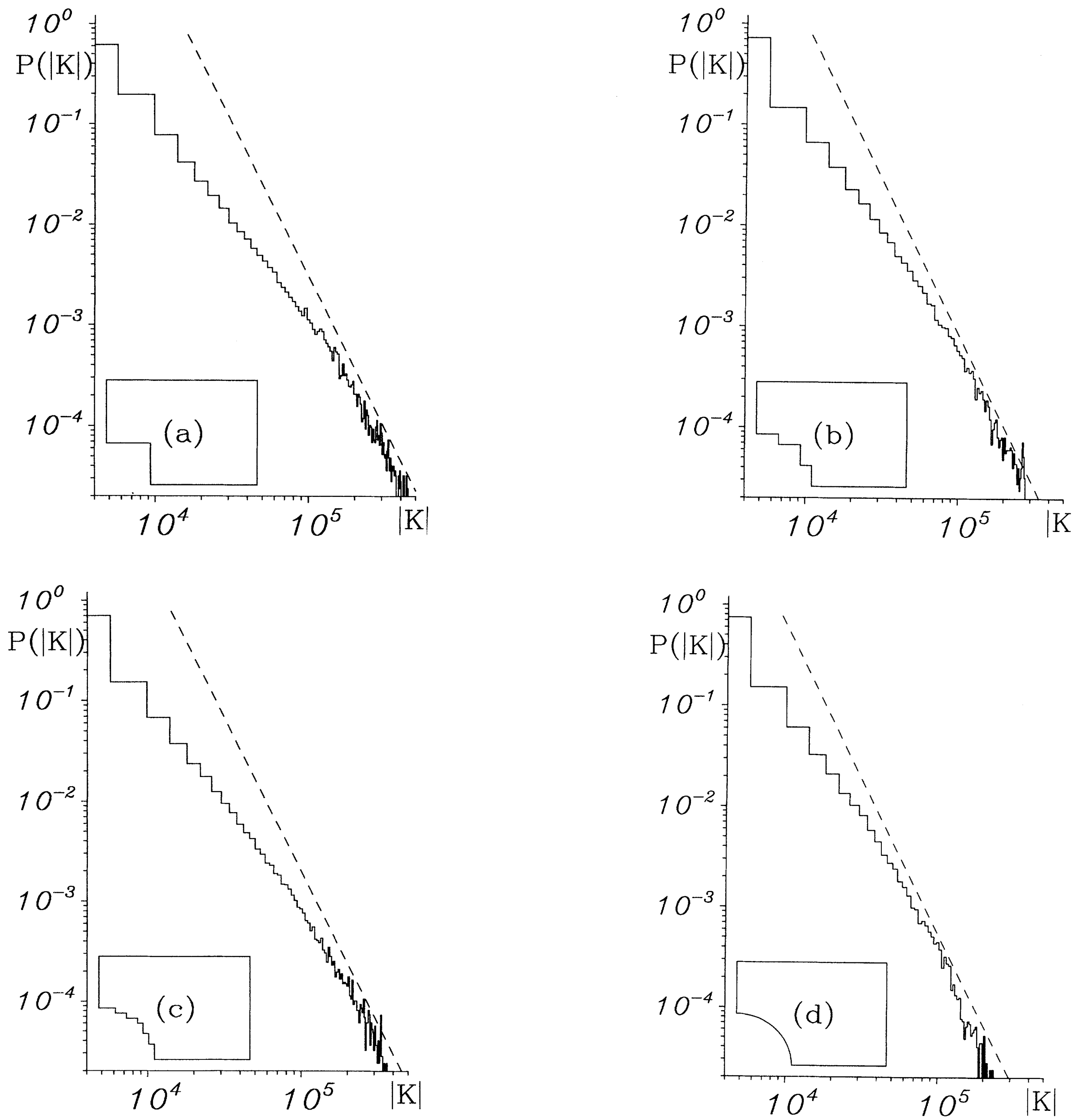

FIG. 1. Histogram of the absolute value of the curvature distribution in logarithmic scale for the two-, four-, and seven-step billiard (a)–(c) and the Sinai billiard (d). The slope -3 of the tail for the GOE is indicated by the broken line. (e)–(h) displays the behavior for small curvatures for the same systems as in (a)–(d). The peak at $K = 0$ gets sharper going from the two-step billiard (e) to the Sinai billiard (h). The abscissa values in (e)–(h) are to be multiplied by 10^3.

of eigenvalues and eigenfunctions up to the 1000th state. The Schrödinger equation is solved in a nonperturbative way by a grid method with asymmetric sparse matrix. A test has shown that 1000 precise eigenfrequencies of microwave cavities [24] agree with calculated eigenvalues for at least four digits.

A. Eigenvalue motion and curvature distribution

To investigate the curvature distribution for pseudointegrable and chaotic billiard systems, we perform numerical studies with four billiard systems: The Sinai billiard [see Fig. 1(d)] and three systems of the shape of two, four, and seven steps, respectively, which are intended as polygonal approximations to the Sinai billiard [3]. The latter are classically nonergodic, whereas the first is known to be a strongly chaotic system [25]. A quantum billiard system—a free particle of mass m moving in a two-dimensional compact domain D—is described by the stationary Schrödinger equation

$$\{-(\hbar^2/2m)\Delta + V(x,y)\}\Psi_n(x,y) = E_n\Psi_n(x,y) \quad (5)$$

with vanishing potential inside D and $V = \infty$ elsewhere.

We choose approximately irrational ratios of the side lengths of the billiards to avoid degeneracies in the pseudointegrable case. For our investigations we calculate the

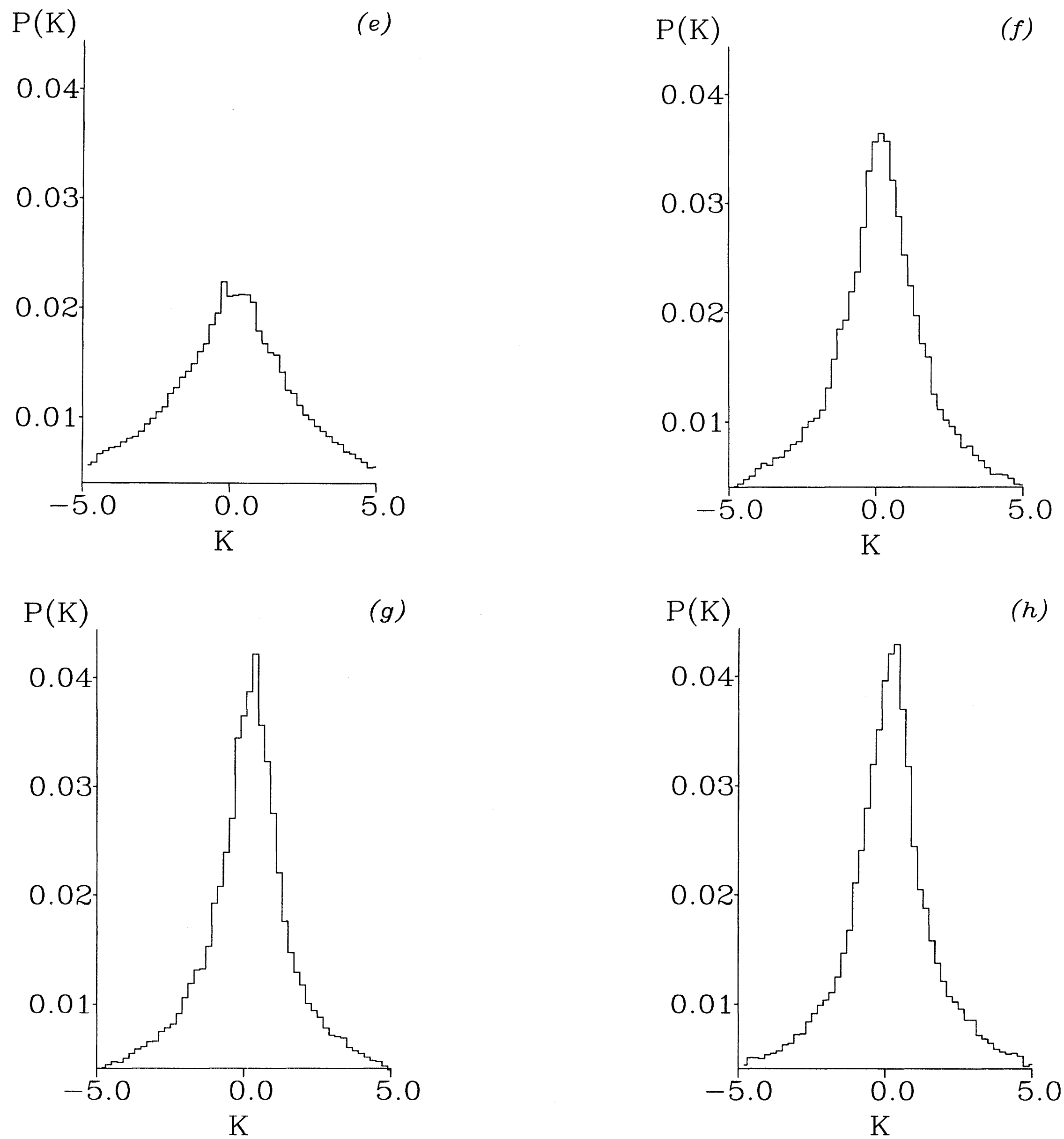

FIG. 1. (*Continued*).

lowest 350 − 400 eigenstates of all systems. These states are the first 350 − 400 eigenstates of odd-odd parity of the "whole" systems, which would be the billiard with a circular or polygonal obstacle in the center of a rectangle. Desymmetrization is necessary as parity is a good quantum number and would lead to degeneracies in the spectrum. However, the statistical theory of eigenvalues applies to Hamiltonians without such symmetries.

A matrix of order 2700 is diagonalized for each billiard shape for 120 geometry parameter values (i.e., the radius of the circular obstacle or its rectangular approximations). The eigenvalues are rescaled to a mean level spacing of unity [20].

For the two-step and the Sinai billiard, the motion of the 300th to 400th eigenvalue of these systems is displayed in Fig. 2. It is obvious that the two-step billiard tends towards an integrable system and the corresponding quantum phenomenon of level clustering [11]: In contrast to the Sinai billiard large areas free of eigenvalues can be seen in the plot of the eigenvalue motion, which means, on the other hand, that more levels get close to each other. This shows the non-neglectible Poissonian part in the spacing distribution of the pseudointegrable system [3].

We have calculated eigenfunctions along several of the striking straight lines in Fig. 2. They often stem from bouncing ball states [26] (see also Fig. 3). As an example the eigenfunctions along the straight line from eigenvalue 366 at radius 0.400 (notation: 0.400,366) to 0.519,340 have been calculated [Fig. 2(b)]. It turns out that a bouncing ball structure moves from 0.400,366 to 0.470,353 and dissolves into irregular structures after the following ACs. This can be interpreted as the occurrence of strong mixing in this region. The scarred eigenfunction then appears again at about 0.491,346 and moves down to 0.519,340.

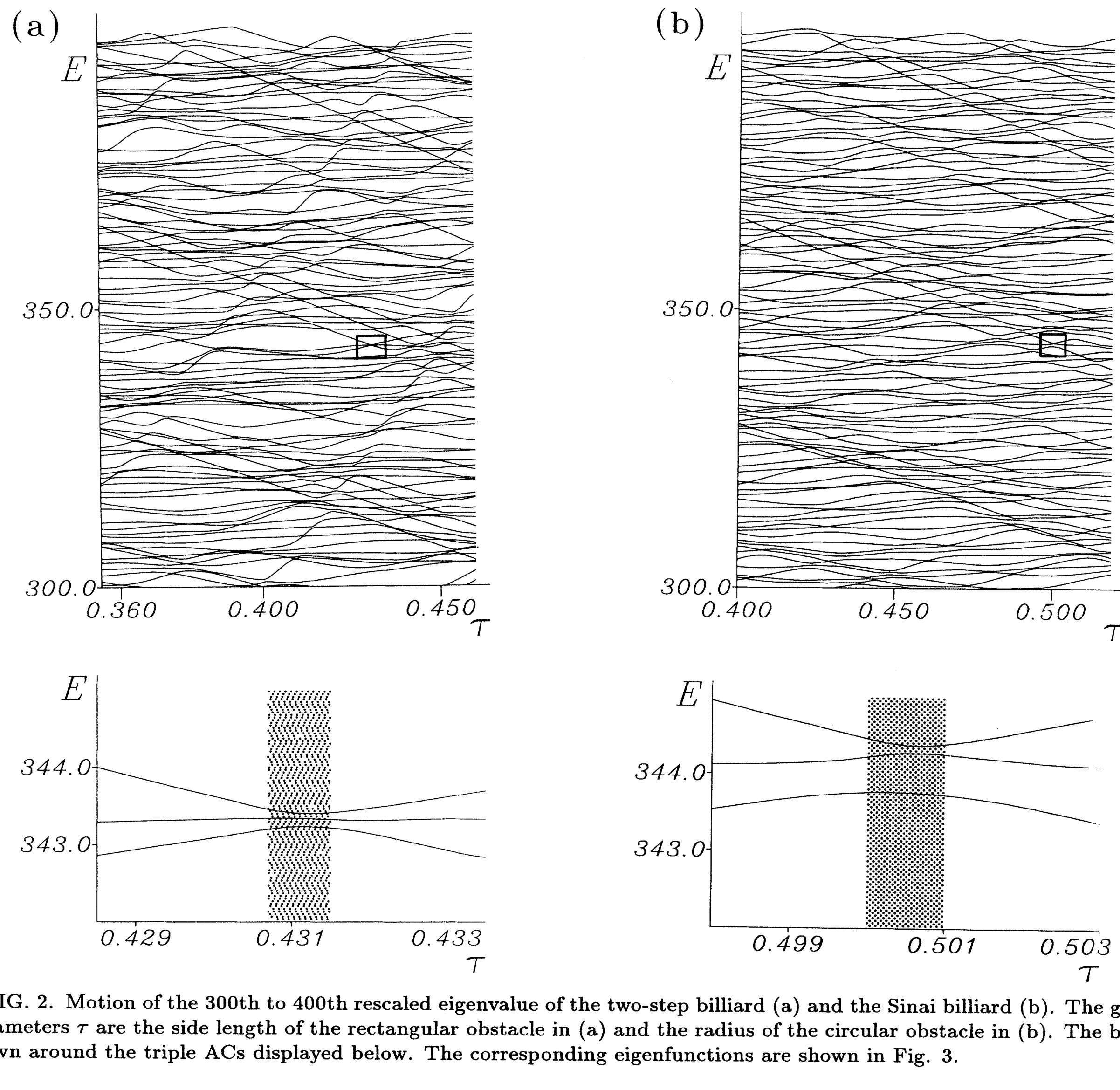

FIG. 2. Motion of the 300th to 400th rescaled eigenvalue of the two-step billiard (a) and the Sinai billiard (b). The geometry parameters τ are the side length of the rectangular obstacle in (a) and the radius of the circular obstacle in (b). The boxes are drawn around the triple ACs displayed below. The corresponding eigenfunctions are shown in Fig. 3.

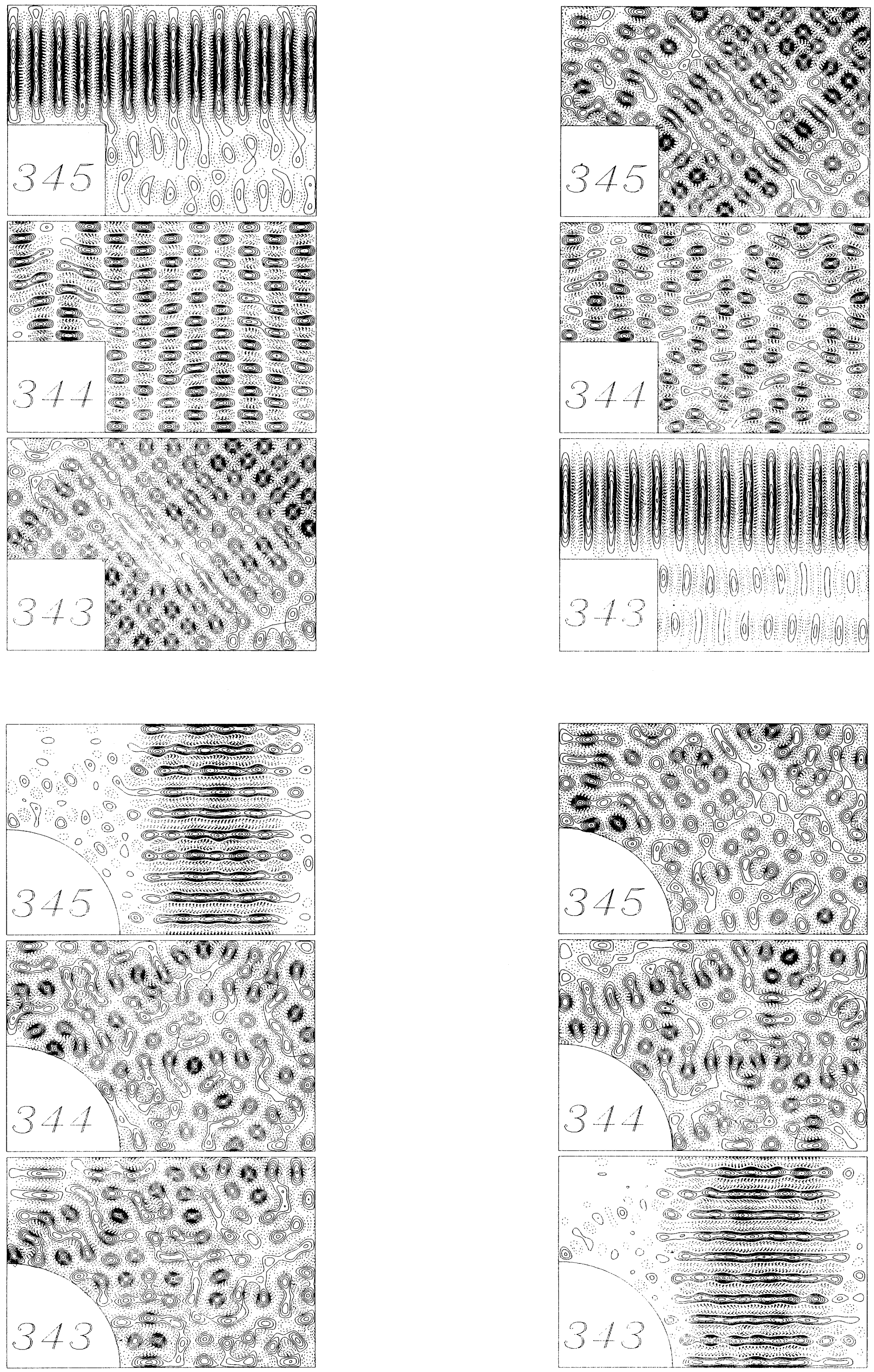

FIG. 3. Plot of the eigenfunctions 343–345 of the two-step billiard at parameter value 0.430 and 0.432 (upper) and the Sinai billiard at parameter value 0.499 and 0.503 (lower), i.e., before and after a triple AC (see the lower part of Fig. 2).

TABLE I. Contributions to the curvature distribution in different ranges of curvature (in %) for the different billiard shapes.

Curvature	$0-1$	$1-10$	$10-10^2$	10^2-10^3	10^3-10^4	10^4-10^5	$>10^5$
two-step	0.02	0.21	2.10	19.81	56.21	19.98	1.68
four-step	0.04	0.34	3.48	28.93	51.06	15.56	0.59
seven-step	0.03	0.34	3.43	29.97	48.83	16.08	1.08
Sinai	0.05	0.38	4.27	32.29	48.50	14.23	0.27

To obtain the curvature distribution we improve our data via interpolation, so that the histograms (Fig. 1) are based on $2.5 \cdot 10^5$ curvature values each. In order to get precise values for the second derivatives of the eigenvalues a B-spline approximation of the $E_i(\tau)$ is performed. The tails of the histograms for the four systems show the behavior predicted for GOE systems (see Fig. 1). It can be seen that for the pseudointegrable systems more high curvatures occur. This may be due to the fact that levels can get very close before repelling each other.

For a more quantitative investigation of the prediction of Eq. (3) a least squares fit of the tails (in logarithmic scale) is performed. The slopes are calculated for different $|K|$ intervals. The results approach values around -3 before getting senseless in the region of very high curvatures. The mean value is -2.91 ± 0.10 for the two-step, -2.98 ± 0.12 for the four-step, -2.78 ± 0.11 for the seven-step billiard, and -3.07 ± 0.24 for the Sinai billiard. This corresponds to linear level repulsion for all systems. Among the different proposals such as how to interpolate between the spacing distributions of integrable systems (Poisson-like) and chaotic systems (Wigner-like) [6,27–29], our result agrees with the one which predicts linear repulsion [6,27].

However, there is a difference between the pseudointegrable and the chaotic systems at small curvature [Fig. 1(e)–1(h)]. The distinct peak around $K = 0$ gets monotonically sharper going from the two-step billiard to the Sinai billiard. Table I shows the same result: There are more small curvatures for the Sinai system and more large curvatures for the two-step system. In [3] such a monotonical transition has been observed for the spacing distribution and the spectral rigidity (Dyson-Mehta statistics).

The nonuniversal behavior in the vicinity of $K = 0$ has been reported in [23] for the stadium and in [15] for the hydrogen atom in a magnetic field. This behavior has been explained with the occurrence of strongly scarred states, which can be related to the straight lines in the parametric spectrum (see above). They give a large contribution at zero curvature.

One would expect that the larger number of bouncing-ball states in the pseudointegrable systems, especially in the two-step billiard, would cause a lot of low curvatures. But the opposite is true: The classically chaotic Sinai billiard has the strongest peak at $K = 0$. This result demands further theoretical investigation.

B. Triple level crossing

As a by-product of our calculations, eigenfunctions at a triple avoided crossing have been investigated. In Fig. 2 many ACs are displayed. Among the many ACs of two levels, some with *three* levels occur. The eigenfunctions 343,344,345 of both the two-step billiard and the Sinai billiard in the vicinity of such a triple AC are shown in Fig. 3. This is an example of the strong interaction of levels belonging to E_i and E_{i+2}.

The regular shapes of the eigenfunction 344 for the two-step billiard remind one of the checkerboard structures of eigenfunctions of a simple rectangular box. Obviously the structures of the upper and the lower levels exchange, whereas the middle one keeps its shape—which is verified by subtraction of the eigenfunctions. At the parameter value where the levels are closest, strong mixing occurs. No clear structures can be seen, except for a very small parameter region for the two-step billiard, where the bouncing ball can be found in the middle level. However, the mixing is confined to a rather small region: For the two-step billiard it is sufficient to change the side length of the obstacle from 0.4307 to 0.4315 to see the transition of the bouncing-ball state, and for the Sinai billiard we have to change the radius of the circular obstacle from 0.5000 to 0.5010, which corresponds to a square side length of 0.4331 and 0.4440, respectively (see the shaded area in the lower part of Fig. 2).

IV. CONCLUSION

In this paper, we have calculated the curvature distribution for pseudointegrable and chaotic billiard systems. The tails of all distributions show the behavior predicted for GOE systems. This agrees with the results for the spacing distribution in such systems, and thus shows the consistency of the theoretical background. However, the occurrence of GOE fluctuations in the spectra of pseudointegrable systems is not yet understood. Extended calculations for the curvature distribution in the semiclassical regime may be necessary. Significant differences for the systems show up at low curvatures. The curvature distribution shows a prominent peak at zero curvature, which is smaller for the pseudointegrable systems. A discussion of the $\hbar$ dependence of the curvature distribution to elucidate the role of bouncing-ball states would be desirable, but was not aimed at in our work. As an aside we have selected triple AC structures in the spectra and investigated the behavior of the eigenfunctions in the vicinity of such avoided crossings.

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